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一个具有零特征的线性双曲型方程组的精确能控性

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摘要: 考虑了一个含有 3 个线性方程且具有零特征的双曲型方程组的精确能控性. 首先,给出了实现精确边界能控的必要条件. 在此基础上,仅通过边界控制(单侧或双侧)即可实现精确能控性. 随后,又讨论了在给出的条件不满足的情况下,通过在非零特征对应的方程上施加适当的内部控制连同边界控制(单侧或双侧)实现了此方程组的精确能控性.

关键词: 线性双曲型方程组;精确能控性;混合初边值问题;边界控制和内部控制

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非线性双曲型方程组的精确能控性结果参见文献[1]. 对于线性双曲型方程组,已经有非常系统的理论和方法[2-3]. 半线性双曲型方程组的结果参见文献[4-6]. 对于一阶拟线性的情形,文献[7]给出了系统的方法和结果,其他结果参见文献[8]. 二阶的拟线性双曲型方程组的精确边界能控性的结果参见文献[9]. 对于具有零特征的双曲型方程组的精确能控性,目前只有一些零零散散的结果发表,给出的结论也不系统. 文献[10]首先证明了含有零特征的双曲型方程组的混合初边值问题在所考虑区域上的 C^1 解(半整体 C^1 解)的存在性及唯一性,然后又实现了相应问题精确能控性. 随后,文献[11]建立了一类具有零特征的拟线性双曲型方程组的精确能控性. 文献[12]对一阶拟线性双曲型方程组在更短的时间内通过内部控制实现了相关问题精确能控性. 文献[13]考虑了一个具有零特征的一阶线性双曲型方程组通过边界和内部控制实现了此类方程的精确能控性.

本文讨论了一个含有零特征的线性双曲型方程组的精确能控性. 主要分三部分:第二部分给出了实现精确边界能控性的必要条件,在此条件满足的情况下,仅通过边界控制就能实现精确能控性. 第三部分,我们讨论了在上述条件不满足的情况下,通过内部控制以及边界控制实现了该问题在所考虑区域上的精确能控性.

1　边界能控

考虑如下具有零特征的线性双曲型方程组

$$\begin{cases} \dfrac{\partial u}{\partial t} = w + v, \\ \dfrac{\partial v}{\partial t} - \dfrac{\partial v}{\partial x} = 0, \\ \dfrac{\partial w}{\partial t} + \dfrac{\partial w}{\partial x} = 0, \end{cases} \tag{1}$$

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给定如下初始条件

$$t=0:(u,v,w)=(u_0,v_0,w_0),\ 0\leqslant x\leqslant L, \tag{2}$$

边界条件

$$x=0: v+w=h(t), \tag{3}$$

$$x=L: v-w=H(t) \tag{4}$$

和终端条件

$$t=T:(u,v,w)=(u_T,v_T,w_T),\ 0\leqslant x\leqslant L. \tag{5}$$

首先，给出了实现精确边界能控的必要条件，并在此基础上实现了上述混合初边值问题的双侧边界能控性和单侧精确边界能控性.

定理1(边界能控的必要条件) 若混合初边值问题(1)~(5)仅通过边界控制即可实现精确能控性，则(u_0,v_0,w_0)和(u_T,v_T,w_T)满足

$$u'_T(x)-v_T(x)+w_T(x)=u'_0(x)-v_0(x)+w_0(x). \tag{6}$$

证明 假设(u,v,w)为混合初边值问题(1)~(4)的满足终端条件(5)的解. 将方程组(1)的第一个方程关于x求导得

$$\frac{\partial^2 u}{\partial t\partial x}=w_x+v_x=-w_t+v_t. \tag{7}$$

将式(7)在区间$[0,T]$上关于t积分得

$$u_x(T,x)-u_x(0,x)=-w(T,x)+w(0,x)+v(T,x)-v(0,x),$$

即

$$u'_T(x)-v_T(x)+w_T(x)=u'_0(x)-v_0(x)+w_0(x).$$

则式(6)成立.

定理2(双侧边界控制) 设$T\geqslant L$，对任何给定的初始条件$(u_0,v_0,w_0)\in C^2[0,L]\times C^1[0,L]\times C^1[0,L]$，终端条件$(u_T,v_T,w_T)\in C^2[0,L]\times C^1[0,L]\times C^1[0,L]$满足条件(6)，则存在边界控制$h\in C^1[0,T]$，$H\in C^1[0,T]$，使得混合初边值问题(1)~(4)的$C^2\times C^1\times C^1$解$(u,v,w)$满足终端条件(5).

证明 令$\tilde{u}=\tilde{u}(t,x)$为如下二阶线性方程组的后向混合初边值问题的解，

$$\begin{cases}\tilde{u}_{tt}=\tilde{u}_{xx}=v'_0(x)-u''_0(x)-w'_0(x),\\ t=T:(\tilde{u},\tilde{u}_t)=(u_T,w_T+v_T),\\ x=0:\tilde{u}=\tilde{h}(t),\\ x=L:\tilde{u}=\tilde{h}(t).\end{cases} \tag{8}$$

其中，$\tilde{h}(t)$和$\tilde{H}(t)$是人为给定的边界条件且分别在$(T,0)$和(T,L)点分别满足C^2相容性条件.

由文献[3]可知，对任何$T\geqslant L$，存在边界控制$\hat{h}\in C^1[0,T]$，$\hat{H}\in C^1[0,T]$，使得前向混合初边值问题

$$\begin{cases}\hat{u}_{tt}-\hat{u}_{xx}=0,\\ t=0:\ \hat{u}=u_0(x)-\tilde{u}(0,x),\\ \qquad\ \hat{u}_t=w_0(x)+v_0(x)-\tilde{u}_t(0,x),\\ x=0:\hat{u}=\hat{h}(t),\\ x=L:\hat{u}=\hat{H}(t)\end{cases} \tag{9}$$

的C^2解$\hat{u}=\hat{u}(t,x)$满足零终端条件

$$t=T:\hat{u}=0,\hat{u}_t=0. \tag{10}$$

由式(8)和(9)，可得$u=\tilde{u}+\hat{u}\in C^2$，且满足

$$\begin{cases}u_{tt}-u_{xx}=v'_0(x)-w'_0(x)-u''_0(x),\\ t=0:u=u_0(x),u_t=w_0(x)+v_0(x),\\ x=0:u=\hat{h}(t)+\tilde{h}(t),\\ x=L:u=\hat{H}(t)+\tilde{H}(t),\\ t=T:u=u_T(x),u_t=w_T(x)+v_T(x).\end{cases} \tag{11}$$

令

$$w = \frac{1}{2}\left(\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} + u'_0 - v_0 + w_0\right),\ v = \frac{1}{2}\left(\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} + u'_0 + v_0 - w_0\right). \tag{12}$$

显然，(u,v,w) 满足方程组(1)的第一个方程. 由式(12)得

$$\frac{\partial v}{\partial t} - \frac{\partial v}{\partial x} = \frac{1}{2}\left(u_{tt} + u_{tx} - u_{xx} - u_{tx} + u''_0(x) - v'_0(x) + w'_0(x)\right) = 0, \tag{13}$$

$$\frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} = \frac{1}{2}\left(u_{tt} - u_{tx} + u_{tx} - u_{xx} + u''_0(x) - v'_0(x) + w'_0(x)\right) = 0. \tag{14}$$

从而方程组(1)满足. 又

$$\begin{cases} t = 0: v = \frac{1}{2}\left(u'_0 + u_t(0,x) - u'_0 + v_0 - w_0\right) = \\ \qquad \frac{1}{2}\left(u'_0 + w_0 + v_0 - u'_0 + v_0 - w_0\right) = v_0, \\ t = 0: w = \frac{1}{2}\left(u_t(0,x) + - u'_0 + u'_0 - v_0 + w_0\right) = \\ \qquad \frac{1}{2}\left(w_0 + v_0 - u'_0 + u'_0 - v_0 + w_0\right) = w_0. \end{cases} \tag{15}$$

$$\begin{cases} t = T: v = \frac{1}{2}\left(u'_T + w_T + v_T - u'_0 + v_0 - w_0\right) = \\ \qquad \frac{1}{2}\left(u'_T + w_T + v_T - u'_T + v_T - w_T\right) = v_T, \\ t = T: w = \frac{1}{2}\left(u_t(T,x) - u'_T + u'_0 - v_0 + w_0\right) = \\ \qquad \frac{1}{2}\left(w_T + v_T - u'_T + u'_T - v_T + w_T\right) = w_T. \end{cases} \tag{16}$$

令

$$h(t) = v(t,0) + w(t,0),\ H(t) = v(t,L) - w(t,L), \tag{17}$$

则 $h \in C^1[0,T]$，$H \in C^1[0,T]$，且 $(u,v,w) = (u(t,x), v(t,x), w(t,x))$ 为混合初边值问题(1)～(4)的解且满足终端条件(5).

定理 3（$x = L$ 处单侧控制）　设 $T \geqslant 2L$，对任何给定的初始条件 $(u_0,v_0,w_0) \in C^2[0,L] \times C^1[0,L] \times C^1[0,L]$，终端条件 $(u_T,v_T,w_T) \in C^2[0,L] \times C^1[0,L] \times C^1[0,L]$ 满足式(6)，任意边界函数 $h \in C^1[0,T]$ 在 $(0,0)$ 和 $(T,0)$ 处分别满足相容性条件，则存在边界控制 $H \in C^1[0,T]$，使得混合初边值问题(1)～(4)的 $C^2 \times C^1 \times C^1$ 解 (u,v,w) 满足终端条件(5).

证明　令 $\tilde{u} = \tilde{u}(t,x)$ 为二阶线性方程组的后向混合初边值问题(8)的解. 对任何 $T \geqslant 2L$，存在边界控制 $\hat{H} \in C^1[0,T]$，使得前向混合初边值问题

$$\begin{cases} \hat{u}_{tt} - \hat{u}_{xx} = 0, \\ t = 0: \hat{u} = u_0(x) - \tilde{u}(0,x),\ \hat{u}_t = w_0(x) + v_0(x) - \tilde{u}_t(0,x), \\ x = 0: \hat{u} = \hat{h}(t) - \tilde{h}(t), \\ x = L: \hat{u} = \hat{H}(t) \end{cases} \tag{18}$$

的 C^2 解 $\hat{u} = \hat{u}(t,x)$ 满足零终端条件(10). 其中，$\hat{h}(t) = u_0(0) + \int_0^t h(t)\,\mathrm{d}t$. 则 $u = \tilde{u} + \hat{u} \in C^2$ 满足

$$\begin{cases} u_{tt} - u_{xx} = v'_0(x) - w'_0(x) - u''_0(x), \\ t = 0:\ u = u_0(x),\ u_t = w_0(x) + v_0(x), \\ x = 0: u = \hat{h}(t), \\ x = L: u = \hat{H}(t) + \tilde{H}(t), \\ t = T: u = u_T(x),\ u_t = w_T(x) + v_T(x). \end{cases} \tag{19}$$

w,v 由式(12)定义. 显然(u,v,w)满足方程组(1)、初始条件(3)和终端条件(5). 而由 $u(t,0)=\hat{h}(t)$,则可得

$$x=0:v+w=\frac{\partial u}{\partial t}=\hat{h}'(t)=h(t),$$

则 $x=0$ 上的边界条件满足.

令

$$H(t)=v(t,L)-w(t,L), \tag{20}$$

则 $H\in C^1$ 即为所求的控制.

2 边界-内部能控性

在条件(6)不满足的情况下,仅通过边界控制无法实现上述混合初边值的精确边界能控性. 于是将内部控制施加到非零特征对应的方程,并通过适当的边界控制(单侧或双侧),实现了上述混合初边值问题的边界-内部精确能控性.

在方程组(1)的第二个和第三个方程中分别施加内部控制 $p(t,x)$ 和 $q(t,x)$,将寻找如下方程组

$$\begin{cases}\dfrac{\partial u}{\partial t}=w+v,\\ \dfrac{\partial v}{\partial t}-\dfrac{\partial v}{\partial x}=p(t,x),\\ \dfrac{\partial w}{\partial t}+\dfrac{\partial w}{\partial x}=q(t,x),\end{cases} \tag{21}$$

以及初始条件(2)和边界条件(3)~(4)的混合初边值问题满足终端条件(5)的解.

定理4(双侧边界内部控制) 设 $T\geqslant L$,对任何给定的初始条件$(u_0,v_0,w_0)\in C^2[0,L]\times C^1[0,L]\times C^1[0,L]$,终端条件$(u_T,v_T,w_T)\in C^2[0,L]\times C^1[0,L]\times C^1[0,L]$,必存在内部控制 $p(t,x)\in C^1[0,T]\times[0,L]$,$q(t,x)\in C^1[0,T]\times[0,L]$ 和边界控制 $h(t)\in C^1[0,T]$,$H(t)\in C^1[0,T]$,使得混合初边值问题(21),(2)~(4)的 $C^2\times C^1\times C^1$ 解(u,v,w)精确地满足终端条件(5).

证明 令 $\tilde{u}=\tilde{u}(t,x)$ 为如下二阶线性波动方程的后向混合初边值问题的解

$$\begin{cases}\tilde{u}_{tt}-\tilde{u}_{xx}=v'_0(x)-w'_0(x)-u''_0(x)+\\ \qquad\dfrac{t}{T}[v'_T(x)-w'_T(x)-u''_T(x)-(v'_0(x)-w'_0(x)-u''_0(x))],\\ t=T:(\tilde{u},\tilde{u}_t)=(u_T,w_T+v_T),\\ x=0:\tilde{u}=\tilde{h}(t),\\ x=L:\tilde{u}=\widetilde{H}(t).\end{cases} \tag{22}$$

其中:$\tilde{h}(t)$ 和 $\widetilde{H}(t)$ 是人为给定的边界条件且在$(T,0)$和(T,L)分别满足 C^2 相容性条件.

对任何 $T\geqslant L$,存在边界控制 $\hat{h}\in C^1[0,T]$,$\hat{H}\in C^1[0,T]$,使得前向混合初边值问题(9)的 C^2 解 $\hat{u}=\hat{u}(t,x)$ 满足零终端条件(10).

令 $u=\tilde{u}+\hat{u}\in C^2$,则 u 满足

$$\begin{cases}u_{tt}-u_{xx}=v'_0(x)-w'_0(x)-u''_0(x)+\\ \qquad\dfrac{t}{T}[v'_T(x)-w'_T(x)-u''_T(x)-(v'_0(x)-w'_0(x)-u''_0(x))],\\ t=0:u=u_0(x),\ u_t=w_0(x)+v_0(x),\\ x=0:u=\hat{h}(t)+\tilde{h}(t),\\ x=L:u=\hat{H}(t)+\widetilde{H}(t),\\ t=T:u=u_T(x),u_t=w_T(x)+v_T(x).\end{cases} \tag{23}$$

令

$$\begin{cases} w = \dfrac{1}{2}\left\{\dfrac{\partial u}{\partial t} - \dfrac{\partial u}{\partial x} - v_0(x) + u'_0(x) + w_0(x) - \dfrac{t}{T}\left[v_T(x) - u'_T(x) - w_T(x) - \left(v_0(x) - u'_0(x) - w_0(x)\right)\right]\right\}, \\ v = \dfrac{1}{2}\left\{\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} + v_0(x) - u'_0(x) - w_0(x) + \dfrac{t}{T}\left[v_T(x) - u'_T(x) - w_T(x) - \left(v_0(x) - u'_0(x) - w_0(x)\right)\right]\right\}. \end{cases} \tag{24}$$

显然,(u,v,w) 满足方程组(21)的第一个方程. 且有

$$\begin{aligned} v_t - v_x = {} & \frac{1}{2}\Big\{u_{tt} + u_{xx} + \frac{1}{T}\left[v_T(x) - u'_T(x) - w_T(x) - \left(v_0(x) - u_0(x) - w_0(x)\right)\right] - \\ & u_{tx} - u_{xx} - v'_0(x) + u''_0(x) + w'_0(x) - \\ & \left(\frac{t}{T}\left[v'_T(x) - u''_T(x) - w'_T(x) - \left(v'_0(x) - u''_0(x) - w_0(x)\right)\right]\right\} = \\ & \frac{1}{2T}\left[v_T(x) - u_T(x) - w_T(x) - \left(v_0(x) - u_0(x) - w_0(x)\right)\right] = p(t,x). \end{aligned}$$

$$\begin{aligned} w_t + w_x = {} & \frac{1}{2}\Big\{u_{tt} - u_{tx} - \frac{1}{T}\left[v_T(x) - u'_T(x) - w_T(x) - \left(v_0(x) - u'_0(x) - w_0(x)\right)\right] + \\ & u_{tx}(x) - u_{xx}(x) - v'_0(x) + u'_0(x) + w'_0(x) - \\ & \frac{t}{T}\left[v'_T(x) - u''_T(x) - w'_T(x) - \left(v_0(x) - u''_0(x) - w'_0(x)\right)\right]\Big\} = \\ & \frac{1}{2T}\left[v_T(x) - u'_T(x) - w_T(x) - \left(v_0(x) - u'_0(x) - w_0(x)\right)\right] = q(t,x). \end{aligned}$$

由式(15)和(16)可得

$$t = 0: w = w_0(x);\ v = v_0(x),$$
$$t = T: w = w_T(x);\ v = v_T(x).$$

令

$$p(t,x) = \frac{\partial v}{\partial t} - \frac{\partial v}{\partial x} =$$
$$\frac{1}{2T}\left[v_T(x) - u'_T(x) - w_T(x) - \left(v_0(x) - u'_0(x) - w_0(x)\right)\right], \tag{25}$$
$$q(t,x) = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} =$$
$$\frac{1}{2T}\left[v_T(x) - u'_T(x) - w_T(x) - \left(v_0(x) - u'_0(x) - w_0(x)\right)\right], \tag{26}$$

则 $p,q \in C^1[0,T] \times [0,L]$ 为所求的内部控制. $h(t)$,$H(t)$ 由式(17)定义,则 $h,H \in C^1[0,T]$ 即为所求的边界控制.

定理 5(单侧边界内部控制)　设 $T \geqslant 2L$,对任何给定的初始条件 $(u_0,v_0,w_0) \in C^2[0,L] \times C^1[0,L] \times C^1[0,L]$,终端条件 $(u_T,v_T,w_T) \in C^2[0,L] \times C^1[0,L] \times C^1[0,L]$,任意边界函数 $h \in C^1[0,T]$ 在 $(0,0)$ 和 $(T,0)$ 处分别满足相容性条件,必存在内部控制 $p(t,x) \in C^1[0,T] \times [0,L]$,$q(t,x) \in C^1[0,T] \times [0,L]$ 和边界控制 $H(t) \in C^1[0,T]$,使得混合初边值问题(21),(2)~(4)的 $C^2 \times C^1 \times C^1$ 解 (u,v,w) 满足终端条件(5).

证明　令 $\tilde{u} = \tilde{u}(t,x)$ 为二阶线性波动方程的后向混合初边值问题(22)的解. 对任何 $T \geqslant 2L$,存在边界控制 $\hat{H} \in C^1[0,T]$,使得前向混合初边值问题(18)的 C^2 解 $\hat{u} = \hat{u}(t,x)$ 满足零终端条件(10).

令 $u = \tilde{u} + \hat{u} \in C^2$ 则

$$\begin{cases} \tilde{u}_{tt} - \tilde{u}_{xx} = v'_0(x) - w'_0(x) - u''_0(x) + \\ \qquad \dfrac{t}{T}\left[v'_T(x) - w'_T(x) - u''_T(x) - \left(v'_0(x) - w'_0(x) - u''_0(x)\right)\right], \\ t = 0:\ u = u_0(x),\ u_t = w_0(x) + v_0(x), \\ x = 0:\ u = \hat{h}(t), \\ x = L:\ u = \hat{H}(t) + \tilde{H}(t), \\ t = T:\ u = u_T(x),\ u_t = w_T(x) + v_T(x). \end{cases} \tag{27}$$

w,v 由式(24)所定义. 显然,(u,v,w)满足方程组(21)、初始条件(3)和终端条件(5). 而由 $u(t,0)=\hat{h}(t)$,则可得:

$$x=0: v+w=\frac{\partial u}{\partial t}=\hat{h}'(t)=h(t).$$

则 $x=0$ 上的边界条件满足. $p(t,x)$,$q(t,x)$ 由式(25)和(26)所定义即为所求的内部控制. $H(t)$ 由式(20)定义,则 $H\in C^1$ 即为所求的边界控制.

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Exact Controllability of a Linear Hyperbolic Systems with Zero Eigenvalues

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Abstract: We consider the exact controllability of a linear hyperbolic systems with zero eigenvalues. First of all, we give a necessary condition about initial and final values. In this situation, we can realize the controllability just by boundary control. Then, we consider that the condition does not meet the case. We add the internal control to the equations with zero eigenvalues. And we realize the exact controllability by local internal controls acting on equations corresponding to non-zero eigenvalues and boundary controls.

Key words: linear hyperbolic equation; exact controllability; mixed initial-boundary value problem; boundary control and internal control

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